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Power series and the open set condition of Hutchinson

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Abstract
We give a sufficient condition for a uniformly contracting self-similar set to have the separation property. For uniform contractions of the real line: $f_i(x) = cx + a_i$, $i = 1, 2, \ldots, r$, with $0 < c < 1$ and $0 = a_1 < a_2 < \cdots < a_r = 1 - c < 1$, we prove that the open set condition of Hutchinson holds if $q(c) \neq 0$ for all power series $q(x) = \sum_{k=0}^{\infty} d_k x^k$, where $d_k = a_i - a_j$, for some $i, j = 1, 2, \ldots, r$, for all $k = 0, 1, 2, \ldots$, and $d_0 \neq 0$. Finally, we give two examples.

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1. Introduction

In the study of fractal geometry, the first simple examples arise in the following manner: given contracting similitudes f_i, $i = 1, 2, \ldots, r$, on Euclidean R^d there exists a unique nonempty compact set A such that $A = \bigcup_{i=1}^{r} f_i(A)$, a so-called self-similar fractal. The first observation in this context is that the similarity dimension and the Hausdorff dimension of A coincide if the 'pieces' $f_i(A)$ are pairwise disjoint. This result remains true if the pieces have only 'small overlap', which is sometimes called 'just touching' [3–5]. To describe this separation property Hutchinson [6] and Moran [8] gave the following definition: the *open set condition* (OSC) is fulfilled iff there is a nonempty open set V such that the sets $f_i(V)$ are disjoint and contained in V. Since the intersection of V and A may be empty, Lalley [7] strengthened the definition as follows: the *strong open set condition* (SOSC) holds iff furthermore $V \cap A \neq \phi$.

When the OSC is fulfilled we have a desirable situation: all the usual definitions of dimensions such as Hausdorff, similarity, box-counting and packing dimensions give the same value [4]. The similarity dimension is easy to compute, and the Hausdorff dimension is generally applicable and has many useful properties. Moreover, many interesting properties of the self-similar set such as its topology, measure or dimension seem to depend on OSC [1, 2, 4, 7, 9, 10].

The OSC of Hutchinson is the most celebrated condition that ensures that there is not too much overlapping. However appealing this condition may be, it is not easy to check since except for some simple examples the open set V, if it exists, may be almost as exotic as A itself. For this reason various equivalent conditions have been developed. Schief [9] proved that the SOSC and the OSC are both equivalent to $H^\alpha(A) > 0$ where α is the similarity dimension of A, and H^α denotes the Hausdorff measure of this dimension. This result shows that an algebraic condition developed previously by Bandt and Graf [2] is also equivalent to the OSC. At first glance the algebraic condition might seem to be handy but in fact, to the best of our knowledge, it is still not easy to check. That is why we looked for other, more convenient conditions.

In the present paper, we will give an analytic condition for uniformly contracting self-similar sets to have the separation property. In order to show that the condition is useful in practice, we finally give two examples. It has to be mentioned that the question of whether the OSC implies the analytic condition remains open.

This paper is organized as follows. In section 2, we present some preliminaries and prove two lemmas that we will need for our work. Section 3 contains the proof of our main results. Finally, section 4 contains two examples.

2. Preliminaries

Let $f_1, \ldots, f_r$ be contracting similitudes on Euclidean R with *contraction ratios* $c_i \in (0, 1)$, i.e. $f_i(x) = c_i x + a_i$, for all $x \in R$. Then the unique nonempty compact set A with $A = \bigcup_{i=1}^r f_i(A)$ is in general a fractal. This set is called *a self-similar set* and is called a *uniformly contracting self-similar set* whenever $c_1 = c_2 = \cdots = c_r = c$.

We will prove two lemmas for our work. One lemma clarifies the relation between two uniformly contracting self-similar sets, and another gives the compact property of the set of analytic functions.

Lemma 2.1. *Let $g_i : R \to R$ be given by $g_i(x) = cx + b_i$, $i = 1, \ldots, r$, with $b_1 < \cdots < b_r$ and $f_i : R \to R$ be given by $f_i(x) = cx + a_i$, with $a_i = (1-c)\frac{b_i - b_1}{b_r - b_1}$, $i = 1, \ldots, r$. Then the following are true:*

(1) $f_i = S^{-1} \circ g_i \circ S$ where S is a bijective mapping $S(x) = \frac{b_r - b_1}{1-c} x + \frac{b_1}{1-c}$.

(2) Let A and B be the self-similar sets with respect to the similitudes $(f_1, \ldots, f_r)$ and $(g_1, \ldots, g_r)$, respectively. Then $S(A) = B$.

Proof.

(1) It is obvious.

(2) Since $A = \bigcup_{i=1}^r f_i(A)$ and $S \circ f_i = g_i \circ S$, we have

$$S(A) = \bigcup_{i=1}^{r} S \circ f_i(A) = \bigcup_{i=1}^{r} g_i \circ S(A). \tag{2.1}$$

By the uniqueness of self-similar set [4] and $B = \bigcup_{i=1}^r g_i(B)$, we get $S(A) = B$. □

From lemma 2.1. we know that we can work with the self-similar set A instead of B without loss of generality, i.e. we may assume that uniform contractions $f_i(x) = cx + a_i$ with $0 = a_1 < a_2 < \cdots < a_r = 1 - c < 1$. Now let us introduce some more notation necessary to state our main results.

Take the uniform contractions, as already mentioned in lemma 2.1, of the real line: $f_i(x) = cx + a_i$, $i = 1, \ldots, r$, $0 < c < 1$, $0 = a_1 < a_2 < \cdots < a_r = 1 - c < 1$. We define the set D. The set $D \subset (-1, 1)$ is the finite set $\{a_i - a_j : i, j = 1, \ldots, r\}$, which is symmetric with respect to $0 \in D$. Furthermore, we denote by Λ the set of sequences

$$d = (d_0, d_1, d_2, \ldots) \in (D \backslash \{0\}) \times D \times D \times \cdots. \tag{2.2}$$

Finally, let us introduce the set Q of analytical functions $q(x)$ given by

$$q(x) = d_0 + d_1 x + d_2 x^2 + \cdots + d_k x^k + \cdots \tag{2.3}$$

where $d_k = a_i - a_j$, for some $i, j = 1, 2, \ldots, r$, for all $k = 0, 1, 2, \ldots$ and $d_0 \neq 0$.

The proof of our main results makes use of the following lemma.

Lemma 2.2. *If $q(c) \neq 0$ for all $q(x) \in Q$, then there is a number $\rho > 0$ such that $|q(c)| \geqslant \rho$, for all $q(x) \in Q$.*

Proof. First, we define a distance $\delta(X, Y)$ between elements $X = (x_k)_{k=0}^{\infty}$, $Y = (y_k)_{k=0}^{\infty} \in \Lambda$ by

$$\delta(X, Y) = \begin{cases} \exp(-\inf\{k \geqslant 0, x_k \neq y_k\}) & \text{if} \quad X \neq Y \\ 0 & \text{if} \quad X = Y. \end{cases} \tag{2.4}$$

Note that two sequences $X, Y \in \Lambda$ are close to each other, if there exists n such that $x_j = y_j$, $0 \leqslant j \leqslant n - 1$. Now we are going to prove that Q is compact in the uniform topology. Let $I = [a, b] \subset (0, 1)$ be a closed interval such that $c \in I$. We are going to be more precise about the set, Q, of analytical functions: an element $q(x) \in Q$ is a power series $q(x) = \sum_{k=0}^{\infty} d_k x^k$, with $x \in I$ and $d_k \in D$. Moreover, if $q(x) \in Q$, then $d_0 \neq 0$. Let $\Phi : \Lambda \to Q$ be defined by

$$[\Phi(d)](x) = \sum_{k=0}^{\infty} d_k x^k \tag{2.5}$$

for $d \in \Lambda$. By the definition of Q, it follows that Φ is onto. Note that Λ is a compact set in the uniform topology.

If $d, d' \in \Lambda$ are sufficiently near, i.e. there exists sufficiently great integer $n \in N$ such that $d_j = d'_j$, $0 \leqslant j \leqslant n - 1$, then the distance between $\Phi(d)$ and $\Phi(d')$ in the uniform topology is

$$\begin{aligned} |\Phi(d) - \Phi(d')| &= |(d_n - d'_n)x^n + (d_{n+1} - d'_{n+1})x^{n+1} + \cdots| \\ &\leqslant (|d_n| + |d'_n|)x^n + (|d_{n+1}| + |d'_{n+1}|)x^{n+1} + \cdots \\ &\leqslant 2a_r(b^n + b^{n+1} + \cdots) = \frac{2b^n a_r}{1 - b}. \end{aligned} \tag{2.6}$$

This implies that Φ is continuous and therefore the set Q is compact. This concludes the proof. □

3. Main results

In this section we obtain a sufficient condition for a uniformly contracting self-similar set to have the separation property, which is an analytic condition associated with the power series described above. Given r contracting similitudes $f_i(x) = cx + a_i$, $i = 1, 2, \ldots, r$, we may assume, without loss of generality (see lemma 2.1 for explanation), that $0 = a_1 < a_2 < \cdots < a_r = 1 - c < 1$, where c is the uniformly contracting ratio. We have the following theorem.

Theorem 3.1. *Let A be a uniformly contracting self-similar set such that $A = \bigcup_{i=1}^{r} f_i(A)$. If $q(c) \neq 0$ for all $q(x) \in Q$, then $(f_1, \ldots, f_r)$ satisfies the OSC.*

Proof. First of all, we start with a useful construction of A for the sake of completeness. It should be emphasized that the construction is due to [4, 6]. Let $A_0 = [0,\ 1]$. It is a standard result that $f_{i_n} \circ \cdots \circ f_{i_1}(A_0)$ is an interval of length c^n for $i_k \in \{1, 2, \ldots, r\}$, $1 \leqslant k \leqslant n$. The union of all such intervals is denoted by A_n and is called the *n-step* of the construction of A. Furthermore, $f_{i_n} \circ \cdots \circ f_{i_1}(A_0)$ is called an interval of A_n. Therefore we get $A = \bigcap_{n=1}^{\infty} A_n$.

It follows from lemma 2.2. that there exists a number $\rho > 0$ such that $|q(c)| \geqslant \rho$, for all $q(x) \in Q$. Now we choose $n_0 > 0$ such that $\rho > c^n$, whenever $n > n_0$. Consider the n-step A_n of the construction A, for some $n \geqslant n_0$, as described above. We now prove that $f_i(A_n) \cap f_j(A_n) = \phi$, for all $i, j = 1, \ldots, r$, $i \neq j$. Suppose, contrary to the assertion, that $f_i(A_n) \cap f_j(A_n) \neq \phi$ for some pair i, j. This means that there are intervals J, J' of A_n such that $f_i(J) \cap f_j(J') \neq \phi$, from which we have that the distance between the left ends of the intervals $f_i(J)$ and $f_j(J')$ is bounded by c^{n+1}. On the other hand, since the left ends of the intervals J and J' are $\sum_{k=0}^{n} a_k c^k$, $a_k \in \{a_1, \ldots, a_r\}$ and $\sum_{k=0}^{n} a'_k c^k$, $a'_k \in \{a_1, \ldots, a_r\}$, respectively, it follows that this distance is

$$f_i\left(\sum_{k=0}^{n} a_k c^k\right) - f_j\left(\sum_{k=0}^{n} a'_k c^k\right) = (a_i - a_j) + \sum_{k=0}^{n} (a_k - a'_k) c^{k+1}.$$

Note that $(a_i - a_j) \neq 0$. Thus, there exists an analytic function $q(x) = \sum_{k=0}^{n+1} d_k x^k$ such that it is an element of Q and $c^{n+1} \geqslant |q(c)| \geqslant \rho > c^n$, which is a contradiction since $0 < c < 1$.

Let V_i be the $c\varepsilon$-neighbourhood of $f_i(A_n), i = 1, 2, \ldots, r$. The number $\varepsilon > 0$ can be chosen such that the open sets $V_1, \ldots, V_r$ are pairwise disjoint, because the sets $f_1(A), \ldots, f_r(A)$ are compact and pairwise disjoint. Let V be the ε-neighbourhood of A_n. It is clear that $f_i(V) = V_i$, for all i. Moreover, $V \supset V_i$ because $A_n \supset f_i(A_n)$. Hence, $(f_1, \ldots, f_r)$ satisfies the open set condition. □

It is well known that for self-similar sets all the usual definitions of dimensions such as Hausdorff dimension $\dim_H A$, packing dimension $\dim_P A$, box-counting dimension $\dim_B A$ and similarity dimension $\dim_S A$ give the same value when the OSC is fulfilled [4]. Then theorem 3.1 yields the following corollary.

Corollary 3.1. *Under the same hypotheses of theorem 3.1,* $\dim_H A = \dim_P A = \dim_B A = \dim_S A = a < 1$ *and* $H^a(A) > 0$, *where a is the similarity dimension of A and* H^a *denotes the Hausdorff measure of this dimension.*

4. Examples

In this section we will give two examples, which show that the analytic condition is useful in practice. More precisely, we are going to apply corollary 3.1. to two specific examples.

Example 4.1. Let A be the self-similar set with respect to the similitudes $f_i(x) = cx + a_i$, $i = 1, 2, \ldots, r$, where $a_1 = 0$, $a_2 = c + \delta_1$, $a_3 = 2c + (\delta_1 + \delta_2), \ldots, a_r = (r-1)c + \sum_{k=1}^{r-1} \delta_k$ with $0 < c, \delta_k < 1$ and $rc + \sum_{k=1}^{r-1} \delta_k = 1$. Then the Hausdorff dimension of A is $\frac{\log r}{-\log c}$. In particular, the middle third Cantor set F with respect to the similitudes $f_i(x) = \frac{1}{3}x + \frac{2(i-1)}{3}$, $i = 1, 2$, has the Hausdorff dimension $\log 2/\log 3$.

Proof. To prove the example it is enough to check that $(f_1, f_2, \ldots, f_r)$ satisfies the analytic condition described in theorem 3.1. For any $q(x) \in Q$, i.e. $q(x) = \sum_{k=0}^{\infty} d_k x^k$ where

$d_k = a_i - a_j$, for some $i, j = 1, 2, \ldots, r$, for all $k = 0, 1, 2, \ldots$ and $d_0 \neq 0$, recalling that $rc + \sum_{k=1}^{r-1} \delta_k = 1$, we have

$$\begin{aligned}|q(c)| &= |d_0 + d_1 c + d_2 c^2 + \cdots| \geqslant |d_0| - (|d_1|c + |d_2|c^2 + \cdots) \\ &\geqslant (c + \min_{1\leqslant i\leqslant r-1} \delta_i) - ((r-1)c + (\delta_1 + \delta_2 + \cdots + \delta_{r-1}))\frac{c}{1-c} \\ &= (c + \min_{1\leqslant i\leqslant r-1} \delta_i) - ((r-1)c + (1-rc))\frac{c}{1-c} = \min_{1\leqslant i\leqslant r-1} \delta_i > 0.\end{aligned}$$

Consequently, corollary 3.1 gives $\dim_H A = \dim_S A = \frac{\log r}{-\log c}$. □

Example 4.2. Let A be the self-similar set with respect to the similitudes

$$f_i(x) = \frac{1}{2r-1}x + \frac{2(i-1)}{2r-1} \qquad i = 1, 2, \ldots, r.$$

Then for any $x \in A$ we have

$$\lim_{\varepsilon\to 0} \frac{\log H^s(A \cap B(\varepsilon, x))}{\log \varepsilon} = \frac{\log r}{\log(2r-1)} (= \dim_H A = \dim_S A)$$

where $B(\varepsilon, x)$ is the open interval of centre x and radius ε.

Proof. First, let s be the similarity dimension of A. It is easy to see $s = \frac{\log r}{\log(2r-1)}$. By the standard proof, we may get $H^s(A) \leqslant |A|^s$ where $|A|$ is the diameter of A, i.e. the greatest distance between any pair of points in A. Thus there exists $C_1 > 0$ such that for any $x \in A$ and $\varepsilon > 0$

$$\limsup_{\varepsilon\to 0} \frac{H^s(A \cap B(\varepsilon, x))}{(2\varepsilon)^s} \leqslant C_1. \tag{4.1}$$

On the other hand, for any $q(x) \in Q$, i.e. $q(x) = \sum_{k=0}^{\infty} d_k x^k$ where $d_k = \pm\frac{2(i-1)}{2r-1}$ for some $i = 1, 2, \ldots, r$, for all $k = 0, 1, 2, \ldots$ and $d_0 \neq 0$, we have

$$\begin{aligned}\left|q\left(\frac{1}{2r-1}\right)\right| &= \left|d_0 + d_1\frac{1}{2r-1} + d_2\left(\frac{1}{2r-1}\right)^2 + \cdots\right| \\ &\geqslant \frac{2}{2r-1} - \frac{2(r-1)}{2r-1}\left(\left(\frac{1}{2r-1}\right) + \left(\frac{1}{2r-1}\right)^2 + \cdots\right) \\ &= \frac{1}{2r-1} > 0\end{aligned}$$

which implies that $H^s(A) > 0$ from corollary 3.1. This means that there exists $C_2 > 0$ such that for any $x \in A$ and $\varepsilon > 0$

$$\liminf_{\varepsilon\to 0} \frac{H^s(A \cap B(\varepsilon, x))}{(2\varepsilon)^s} \geqslant C_2. \tag{4.2}$$

Finally, the inequality (4.2) together with (4.1) yields

$$\lim_{\varepsilon\to 0} \frac{\log H^s(A \cap B(\varepsilon, x))}{\log \varepsilon} = s = \frac{\log r}{\log(2r-1)}$$

and we conclude the proof of this example. □

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